ADDITIVITY OF CIRCULAR WIDTH

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Abstract. We show that circular width is preserved under connected sum of knots for some cases.

1. Introduction

In [MG] the second author defined *circular thin position* and *circular width* for a knot in S^3. The idea is to find collections of surfaces $\{S_i\}_{i=1}^n$ and $\{F_i\}_{i=1}^n$, not necessarily connected, which are properly embedded in the knot exterior, such that each F_i and each S_i contains a Seifert surface for the knot. When the knot complement is cut open along the collection $\{F_i\}_{i=1}^n$ the result is a collection of disjoint submanifolds whose Heegaard surfaces are the $S_i's$. We assign a complexity $c(S_i)$ to each S_i, and define the circular width of the exterior of the knot, $cw(E(K))$, as the minimal ordered n-tuple that encodes these complexities.

A decomposition that realizes the circular width of the knot is called circular thin position of the knot. Circular thin position guarantees that all the $F_i's$ are incompressible and all the $S_i's$ are weakly incompressible. Hence when the knot complement is in circular thin position we obtain a nice sequence of Seifert surfaces which are alternately incompressible and weakly incompressible.

Given two knots K_1 and K_2 in S^3, we can take their connected sum $K_1\sharp K_2$, it is natural to study the behavior of circular width under this operation. In [MG] an upper bound for the circular width of $K_1\sharp K_2$ is given, which depends on the circular width of the original knot exteriors. Namely;

$$cw(E(K_1\sharp K_2)) \leq cw(E(K_1))\sharp cw(E(K_2)) \tag{1}$$

In this paper we analize knots in S^3 having a circular thin position containing a minimal genus Seifert surface and we prove that equality in equation (1) holds.

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Our main result is:

Theorem 1.1. *Let K_1 and K_2 be knots in S^3. The equation*
$cw(E(K_1\sharp K_2)) = cw(E(K_1))\sharp cw(E(K_2))$
holds for the following cases:

(1) *K_1 and K_2 are fibered knots.*
(2) *K_1 is fibered and K_2 is not fibered.*
(3) *K_1 and K_2 are non-fibered knots. $E(K_1)$ and $E(K_2)$ have circular thin positions containing minimal genus Seifert surfaces as a thin level.*

This paper is organized as follows. In Section 2 we give definitions and some facts about knots and Heegaard splittings. Circular thin position is defined in Section 3, we also discuss the behavior of circular width under connected sum of knots. In Section 4 we study in detail ordered n-tuples, we prove Proposition 4.4 which is a technical result needed to prove our main theorem. In Section 5 we prove Proposition 5.1 and Corollary 5.3 which allow us to construct a circular handle decomposition for each summand in a connected sum of two knots, we also prove Theorem 1.1.

2. Preliminaries

In this section we begin by briefly recalling some notions for the theory of knots, Seifert surfaces and Heegaard splittings.

2.1. **Knots and surfaces.** This section is devoted to definitions related to knots and Seifert surfaces, as well as to properties of Seifert surfaces under two operations on knots. The definitions and operations are mostly classical.

Let K be a knot in S^3. The knot complement will be denoted by $C_K = S^3 \setminus K$. An open tubular neighborhood of K will be denoted by $N(K)$ and the exterior of the knot K by $E(K) = S^3 \setminus N(K)$.

A Seifert surface R' for a knot K is an oriented compact 2-submanifold of S^3 with no closed components such that $\partial R' = K$. The intersection of R' with $E(K)$, $R = R' \cap E(K)$, is also called a Seifert surface for K.

The *genus* of a knot K is the least genus of all its Seifert surfaces. A surface realizing the genus of a knot is called a *minimal genus Seifer surface*.

Since R is two sided we can specify a $+$*side* and a $-$*side* of R. We say that a disk D, such that $\partial D \subset R$, lies on the $+$*side* (resp. in the $-$*side*) of R if the collar of its boundary lies on the $+$*side* (resp. in the $-$*side*) of R.

Definition 2.1. Let S be a surface in a 3-manifold M. We say that S is *compressible* if there is a 2-disk $D \subset M$ such that $D \cap int(S) = \partial D$ does not bound a disk in S. D is a compressing disk for S. If S is not compressible, it is said to be *incompressible*.

We say that S is *strongly compressible* if there are two compressing disks, D_1 lying on the $+$side of S and D_2 lying on the $-$side of S, with ∂D_1 and ∂D_2 disjoint essential closed curves in S. Otherwise we say that S is *weakly incompressible*.

Definition 2.2. The *connected sum of two knots* K_1 and K_2, denoted by $K_1 \sharp K_2$, is constructed by removing a short segment from each K_i and joining each free end of K_1 to a different end of K_2 to form a new knot. This operation is well-defined up to orientation. There is a 2-sphere Σ that intersects $K_1 \sharp K_2$ in two points and decomposes it in K_1 and K_2. Σ is called a *decomposing* sphere.

Given Seifert surfaces S_1 and S_2 for K_1 and K_2, respectively, one may construct a Seifert surface for the knot $K_1 \sharp K_2$ by taking a boundary connected sum of S_1 and S_2, denoted by $S_1 \sharp_\partial S_2$.

2.2. **Heegaard splittings.** All manifolds will be orientable.

Definition 2.3. A *compression body* W is a cobordism rel ∂ between surfaces $\partial_+ W$ and $\partial_- W$ such that $W \cong \partial_+ W \times [0,1] \cup$2-handles$\cup$3-handles, where the 2-handles are attached along $\partial_+ W \times 1$ and any resulting 2-sphere is capped off with a 3-handle. If $\partial_- W \neq \emptyset$ and W is connected, then W is obtained from $\partial_- W \times I$ by attaching a number of 1-handles along disks on $\partial_- W \times \{1\}$ where $\partial_- W$ corresponds to $\partial_- W \times \{0\}$.

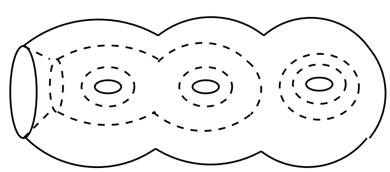


Figure 1. A compression body W with $\partial_- W$ a genus 2 surface with one boundary component and a genus 1 surface. $\partial_+ W$ is a genus 3 surface with one boundary component

Definition 2.4. A *3-manifold triad* $(M; N, N')$ is a cobordism M rel ∂ between surfaces N and N'. Thus N and N' are disjoint surfaces in ∂M with $\partial N \cong \partial N'$ such that $\partial M = N \cup N' \cup (\partial N \times I)$.

Definition 2.5. A *Heegaard splitting* of $(M; N, N')$ is a pair of compression bodies (W, W') such that $W \cup W' = M$, $W \cap W' = \partial_+ W = \partial_+ W' (= S)$ and $\partial_- W = N$, $\partial_- W' = N'$.

S is called a Heegaard surface and $\partial S \cong \partial N$.

The *genus* of a Heegaard splitting is defined by the genus of the Heegaard surface.

A Heegaard splitting (W, W') is said to be *weakly reducible* if there are disks $D_1 \subset W$ and $D_2 \subset W'$ with $\partial D_i \subset S$ an essential curve, for $i = 1, 2$, and such that $\partial D_1 \cap \partial D_2 = \emptyset$.

If the Heegaard splitting is not weakly reducible then it is said to be *strongly irreducible.*

Definition 2.6. A Heegaard splitting $M = H_1 \cup_S H_2$ is ∂-reducible if there is a ∂-reducing disk for M which intersects S in a single curve.

Proposition 2.7. *(see* [S] *Proposition 3.6) Any Heegaard splitting of a ∂-reducible 3-manifold is ∂-reducible.*

3. Circular thin position

This was introduced by the second author in [MG]. For sake of completeness we include some definitions and results.

Given a regular circled-valued Morse function on the complement of a knot C_K, $f : C_K \to S^1$, as in the case of real-valued Morse functions, there is a correspondence between f and a handle decomposition for $E(K)$, namely

$$E(K) = (R \times I) \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup ... \cup N_k \cup T_k \cup b_3 / (R \times 0 \sim R \times 1),$$

where R is a Seifert surface for K, $R \setminus K$ is a regular level surface of f, N_i is a collection of 1-handles corresponding to index 1 critical points, T_i is a collection of 2-handles corresponding to index 2 critical points and b_3 is a collection of 3-handles.

We will call this decomposition a *circular handle decomposition* for $E(K)$.

Let us denote by S_i the surface $cl(\partial((R \times I) \cup N_1 \cup T_1 ... \cup N_i) \setminus \partial E(K) \setminus R \times 0)$ and let F_{i+1} be the surface $cl(\partial((R \times I) \cup N_1 \cup T_1 ... \cup T_i) \setminus \partial E(K) \setminus R \times 0)$, where cl means the closure. When $i = k$, $F_{k+1} = F_1 = R$. Every S_i and F_i contains a Seifert surface for K; note that F_i or S_i may be disconnected.

The surfaces S_i and F_i, for $i = 1, 2, ..., k$ will be called *level surfaces.*

A level surface F_i is called a *thin surface* and a level surface S_i is called a *thick surface.*

Let $W_i =$ (collar of F_i)$\cup N_i \cup T_i$. W_i is divided by a copy of S_i into two compression bodies $A_i =$ (collar of F_i)$\cup N_i$ and $B_i =$ (collar of S_i)$\cup T_i$.

Thus S_i describes a Heegaard splitting of W_i into compression bodies A_i and B_i, where $\partial_- A_1 = R$, $\partial_+ A_i = \partial_+ B_i = S_i$, $\partial_- B_i = \partial_- A_{i+1} = F_{i+1}$ ($i = 1, 2, ..., k-1$), $\partial_- B_k = R$. Thus we can write

$E(K) = A_1 \cup_{S_1} B_1 \bigcup_{F_2} A_2 \cup_{S_2} B_2 \bigcup_{F_3} ... \bigcup_{F_k} A_k \cup_{S_k} B_k$.

Figure 2 shows a schematic picture of a circular handle decomposition with level surfaces and compression bodies indicated.

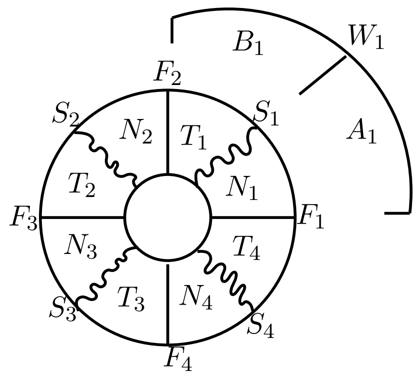


Figure 2. Splitting of $E(K)$ into compression bodies

We wish to find a decomposition in which the S_i are as simple as possible.

Definition 3.1. For a compact connected surface G different from S^2 or D^2 define the complexity of G, $c(G)$, to be $c(G) = 1 - \chi(G)$. If $G = S^2$ or $G = D^2$, set $c(G) = 0$. If G is disconnected we define $c(G) = \Sigma(c(G_i))$ where G_i are the components of G.

Let K be a knot in S^3. Let D be a circular handle decomposition for $E(K)$. Define the *circular width of* $E(K)$ *with respect to the decomposition* D , $cw(E(K), D)$, to be the set of integers $\{c(S_i), 1 \leq i \leq k\}$. Arrange each multi-set of integers in monotonically non-increasing order, and then compare the ordered multisets lexicographically.

The *circular width of* $E(K)$, denoted $cw(E(K))$, is the minimal circular width, $cw(E(K), D)$ over all possible circular decompositions D for $E(K)$.

$E(K)$ is in *circular thin position* if the circular width of the decomposition is the circular width of $E(K)$.

If a knot K is fibered we define the circular width of K, $cw(E(K))$, to be equal to zero.

A nice property of a knot in circular thin position is that the thin surfaces are incompressible and the thick surfaces are weakly incompressible. For a proof of this fact see Theorem 3.2, [MG].

Definition 3.2. A circular handle decomposition D for a knot exterior $E(K)$ is called a *circular locally thin* decomposition if the thin level surfaces F_i's are incompressible and the thick level surfaces S_i's are weakly incompressible.

Definition 3.3. K is *almost fibered* if there is a Seifert surface R so that $E(K)$ has a circular thin decomposition of the form
$E(K) = (R \times I) \cup N_1 \cup T_1/(R \times 0 \sim R \times 1)$.

Figure 3 shows a schematic picture of an almost fibered knot.

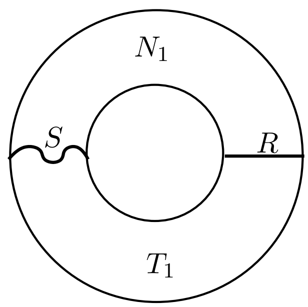


Figure 3. An almost fibered knot.

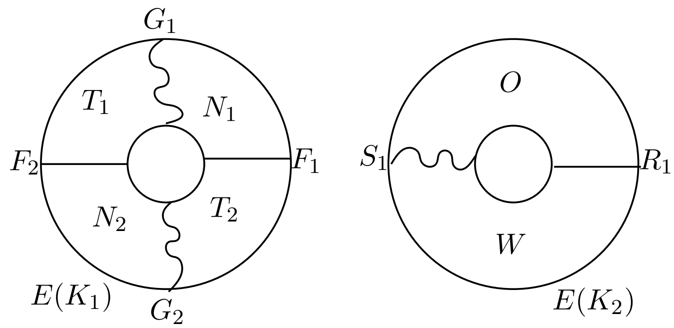
3.1. **Behavior of circular width under connected sum.** Let us consider the knot exteriors $E(K_1)$ and $E(K_2)$. Assume they have the following circular handle decompositions:

$E(K_1) = (F_1 \times I) \cup N_1 \cup T_1 \cup N_2 \cup T_2 \cup ... \cup N_n \cup T_n \cup b_3^1/(F_1 \times 0 \sim F_1 \times 1)$
with level surfaces F_1, G_1, F_2..., F_n, G_n.

$E(K_2) = (R_1 \times I) \cup O_1 \cup W_1 \cup O_2 \cup W_2 \cup ... \cup O_m \cup W_m \cup b_3^2/(R_1 \times 0 \sim R_1 \times 1)$
with level surfaces R_1, S_1, R_2...,R_m, S_m.

Let $K = K_1 \sharp K_2$. There is a natural way to obtain a circular handle decomposition for $E(K)$ as follows. Starting with the Seifert surface $R = F_1 \sharp_\partial R_1$ for K, we attach the sequence of handles corresponding to $E(K_1)$, i.e., we attach N_i and T_i, along the F_1 summand of R. Then we attach the sequence of handles corresponding to $E(K_2)$, i.e., we attach O_j and W_j, along the R_1 component of R. Notice that this process can be done if we choose different thin surfaces. Thus $K_1 \sharp K_2$ inherits $n \times m$ circular handle decompositions each with $n + m$ thin levels and thick levels.

The thin levels for $K = K_1 \sharp K_2$ are homeomorphic to $\{F_{i_0} \sharp R_j\} \cup \{F_i \sharp R_{j_0}\}$ and the thick levels are homeomorphic to $\{F_{i_0} \sharp S_j\} \cup \{G_i \sharp R_{j_0}\}$, for a fixed $i_0 \in \{1, 2, ..., n\}$ and $j_0 \in \{1, 2, ..., m\}$.



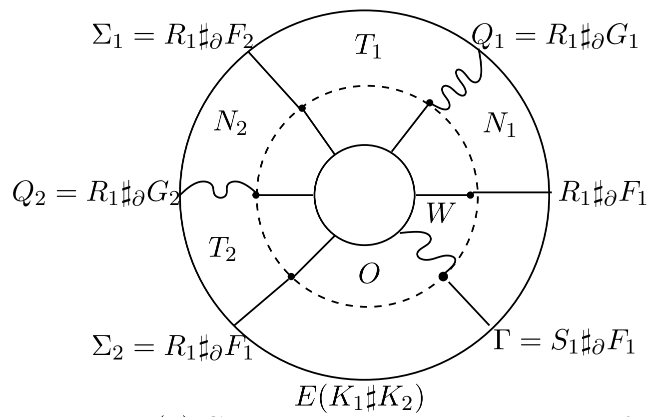


Figure 4. (a) Circular handle decomposition for $E(K_1)$, (b) circular handle decomposition for $E(K_2)$ and (c) induced circular handle decomposition for $E(K_1\sharp K_2)$.

Figure 4 is a schematic picture of the induced circular handle decomposition in a complement of a connected sum of two knots.

Since the Euler characteristic for the boundary connected sum equals $\chi(S_1\sharp_\partial S_2) = \chi(S_1) + \chi(S_2) - 1$, then the complexity $c(S) = 1 - \chi(S)$ applied to a boundary connected sum $S_1\sharp_\partial S_2$ becomes equal to $c(S_1\sharp_\partial S_2) = c(S_1) + c(S_2)$.

Each decomposition for $E(K_1\sharp K_2)$ has circular width:

$$cw_D(E(K_1\sharp K_2)) = \{c(F_{i_0}\sharp S_1), ..., c(F_{i_0}\sharp S_m), c(G_1\sharp R_{j_0}), ...c(G_n\sharp R_{j_0})\}$$

modulo non-increasing order.

If we choose F_{i_0} to be a thin level Seifert surface for K_1 such that $c(F_{i_0}) \leq c(F_i)$ for all $i = 1, 2, ..., n$ and R_{i_0} be a thin level Seifert surface for K_2 such that $c(R_{j_0}) \leq c(R_j)$ for all $j = 1, 2, ..., m$, then the decomposition D for $E(K_1\sharp K_2)$ containing F_{i_0} and R_{j_0} as summands of

the thick levels will be the one with the smallest circular width amongst all the $n \times m$ circular decompositions.

Let us denote the circular width of such decomposition by:

$$cw(E(K_1))\sharp cw(E(K_2))$$

which is an upper bound for $cw(E(K_1 \sharp K_2))$. Thus we have:

$$cw(E(K_1 \sharp K_2)) \leq cw(E(K_1))\sharp cw(E(K_2)) \tag{2}$$

Moreover, in [MG] it is proved that if K_1 and K_2 are in circular thin decomposition, the circular handle decomposition induced on $K_1 \sharp K_2$ is circular locally thin. Thus, is natural to ask if such decomposition is the thinnest for $K_1 \sharp K_2$ and if equality in (2) holds.

4. Ordered n-tuples

We have defined the circular width as an ordered n-tuple, say $a = (a_1, a_2, ..., a_n)$ where $a_1 \geq a_2 \geq ... \geq a_n$. For simplicity it will be called just and n-tuple.

We can compare a m-tuple and a n-tuple using the lexicographic order. More precisely, we have the following definition.

Definition 4.1. Let $a = (a_1, a_2, ..., a_n)$ be a n-tuple and let $b = (b_1, b_2, ..., b_m)$ be a m-tuple. We say that;

(1) $a = b$ if and only if $m = n$ and $a_i = b_i$ for all i.
(2) $a < b$
  (a) If there exists i_o such that $a_{i_o} < b_{i_o}$ and $a_i = b_i$ for all $i < i_o$, or
  (b) If $n < m$ and $a_i = b_i$ for all $1 \leq i \leq n$.

Remark 4.2. If a is a n-tuple and b is a m-tuple such that $a \leq b$ and α a non-negative real number, then $(a_1 + \alpha, ..., a_k + \alpha) \leq (b_1 + \alpha, ..., b_l + \alpha)$.

Given a n-tuple a and a m-tuple b we can define a new $(n+m)$-tuple as follows:

Definition 4.3. Let a be a n-tuple and b be a m-tuple. Define the union of a and b, denoted by $a \cup b$, as the $(n+m)$-tuple whose entries are all the elements of $\{a_1, a_2, ..., a_n, b_1, ..b_m\}$, ordered in non-increasing order.
For instance if $a = (4, 4, 3, 3, 1, 1, 1)$ and $b = (7, 7, 5, 5, 5, 3, 3, 3, 1, 1, 1, 1)$ then $a \cup b = (7, 7, 5, 5, 5, 4, 4, 3, 3, 3, 3, 3, 1, 1, 1, 1, 1, 1, 1)$.

The following result is used in the proof of Theorem 5.6:

Proposition 4.4. *Let a, b, c and d tuples such that $b \leq a$ and $d \leq c$. Then $b \cup d \leq a \cup c$.*

Proof. Case 1: If $a = b = c = d$, it is easy to verify that $a \cup c = b \cup d$.

Case 2: If $a = b$ and $d < c$.

Subcase 2.1: There is an index j_0 such that $d_{j_0} < c_{j_0}$ and $d_s = c_s$ for all $s < j_0$.

Suppose there is l_0 such that $a_{l_0} < c_{j_0} \leq a_{l_0-1}$. Then the $(l_0 + j_0 - 2)$th-entry for both $a \cup c$ and $b \cup d$ coincide. The $(l_0 + j_0 - 1)$th-entry for $a \cup c$ is either c_{j_0} or a_{l_0}, by assumption $a_{l_0} < c_{j_0}$, thus it must be c_{j_0}. On the other hand the $(l_0 + j_0 - 1)$th-entry for $b \cup d$ is chosen from d_{j_0} and a_{l_0}, in either case both are strictly smaller than c_{j_0}, therefore $b \cup d < a \cup c$.

Suppose that $c_{j_0} < a_k$ for all k. If there is l_0 such that $a_{l_0-1} = c_{j_0-1} > a_{l_0}$, then the $(l_0 + j_0 - 2)$th-entry for both $a \cup c$ and $b \cup d$ coincide. The $(l_0 + j_0 - 1)$th-entry for $a \cup c$ is either a_{l_0} or c_{j_0}, by assumption $c_{j_0} < a_k$ for all k then we must choose a_{l_0}, the next entry is a_{l_0+1}, and so on until the entry is a_n, then the entry that follows must be c_{j_0}. Similarly happens for $b \cup d$, its $(l_0 + j_0 - 1)$th-entry is a_{l_0}, the next one is a_{l_0+1}, and so on until the entry is a_n, then the next entry is d_{j_0} which is strictly smaller that c_{j_0}, thus $b \cup d < a \cup c$.

Subcase 2.2: d is a n-tuple and c is a m-tuple such that $n < m$ and $d_i = c_i$ for all $1 \leq i \leq n$. Suppose $a = b$ is a k-tuple.

If $a_j \geq d_n$ for all $1 \leq j \leq k$. Then $a \cup c$ is a $(m + k)$-tuple and $b \cup d$ is a $(n + k)$-tuple such that $n + k < m + k$ and the entries of $a \cup c$ and $b \cup d$ coincide up to the $(n + k)$th-entry. Thus $b \cup d < a \cup c$.

If there is l_0 such that $a_{l_0} < d_n \leq a_{l_0-1}$. Then $a \cup c$ and $b \cup d$ coincide up to the $(n + l_0 - 1)$th-entry which is equal to $c_n = d_n$. The remainder entries for $b \cup d$ are $a_{l_0}, a_{l_0+1}, ..., a_k$ in that order. On the other hand the remainder entries for $a \cup c$ are taken from $\{c_s, n < s \leq m\}$ and $\{a_t, l_0 \leq t \leq k\}$. Then either $a \cup c$ and $b \cup d$ are equal up to the (n+k)th-entry, or there is a $u_0 > n + l_0 - 1$ such that $x_{u_0} < y_{u_0}$ where x_{u_0} is an entry of $b \cup d$ and y_{u_0} is an entry for $a \cup c$, which imply that $b \cup d < a \cup c$.

Case 3: If $b < a$ and $d < c$. Using case 2, we have that $b \cup d < a \cup d$ and that $d \cup a < c \cup a$. These two inequalities imply $b \cup d < a \cup c$. □

5. Additivity of circular width under connected sum

First we need to prove that a circular (locally) thin handle decomposition for $E(K_1 \sharp K_2)$ induces a circular handle decomposition on each summand $E(K_1)$ and $E(K_2)$.

Recall that for a connected sum of knots, $K_1 \sharp K_2$, there is a decomposing sphere Σ that intersects $K_1 \sharp K_2$ in two points. Let A be the annulus in $E(K_1 \sharp K_2)$ given by $\Sigma \cap E(K_1 \sharp K_2)$.

The following proposition shows that A intersects the collection of thin and thick surfaces for $E(K_1 \sharp K_2)$ in essential arcs.

Proposition 5.1. *Suppose that $E(K_1 \sharp K_2)$ is in circular (locally) thin position with $\mathcal{F}$ the family of thin surfaces and $\mathcal{S}$ the family of thick surfaces. Then $\mathcal{F} \cup \mathcal{S}$ can be isotoped to intersect A only in arcs that are essential in both A and $\mathcal{F} \cup \mathcal{S}$.*

Proof. The annulus A is properly embedded in $E(K_1 \sharp K_2)$, its boundary components are meridian disks in $\partial E(K_1 \sharp K_2)$. We arrange A and $\mathcal{F} \cup \mathcal{S}$ to be transverse and conclude that A intersects each $F_i \in \mathcal{F}$ and each $S_i \in \mathcal{S}$ in exactly one arc (properly embedded and essential in A) and a finite number of simple closed curves. We need to remove this later curves.

Let $F_i \in \mathcal{F}$, since F_i is incompressible and $E(K_1 \sharp K_2)$ is irreducible then, using an innermost disk argument, $A \cap F_i$ does not contain closed curves. Thus $F_i \cap A$ consists of a single properly embedded separating essential arc in F_i.

All curves in $A \cap S_i$ are essential in S_i, otherwise using the irreducibility of $E(K_1 \sharp K_2)$ we get rid of of inessential curves.

Each S_i determines a Heegaard splitting given by $A_i \cup_{S_i} B_i$, with $\partial_- A_i = F_i$, $\partial_- B_i = F_{i+1}$ and $\partial_+ A_i = \partial_+ B_i = S_i$. Let R_i be the region on A cobounded by the arcs $\alpha_i = A \cap F_i$ and $\alpha_{i+1} = A \cap F_{i+1}$. R_i contains an arc β_i and simple closed curves contained in $A \cap S_i$.

A disk of $R_i - S_i$ compresses S_i in one of the two compression bodies A_i or B_i, say A_i. Since S_i is weakly incompressible, all disks components of $R_i - S_i$ lie in A_i.

Claim 5.2. The curves in $R_i \cap S_i$ are non nested in R_i.

If any pair of curves of $R_i \cap S_i$ are nested (they are inessential in R_i) then the outer curve of the innermost such pair cuts off a component C of $R_i - S_i$ so that all but one of the curves in ∂C are adjacent to disks in A_i (thus $C \subset B_i$) and precisely one, denoted by γ, is not. Compress S_i into A_i along 2-handles whose cores are the disks with boundaries on ∂C. Let $\bar{S}_i$ be the result of this compression. Let $\bar{B}_i$ be the 3-manifold obtained from B_i by attaching these 2-handles to B_i. Thus S_i determines a Heegaard splitting for $\bar{B}_i$, namely $\bar{B}_i = B_i \cup_{S_i} B'_i$, where $B'_i = (S_i \times I) \cup$2-handles, $\partial_- B_i = F_{i+1}$ and $\partial_- B'_i = \bar{S}_i$. A copy of the curve γ lies in $\bar{S}_i$ and it is the boundary of a disk D in $\bar{B}_i$. Suppose that γ is non-trivial in $\bar{S}_i$ so D is a ∂-reducing disk for $\bar{B}_i$. Then the

Heegaard splitting $\bar{B}_i = B_i \cup_{S_i} B_i'$ is ∂-reducible, by Proposition 2.7 there is a ∂-reducing disk D' for $\bar{B}_i$ that intersects S_i in a single curve γ'. Moreover $\partial D' \subset \bar{S}_i$.

Observe that D' intersects B_i' in an annulus A' with one boundary component on $\bar{S}_i$ and the other one on S_i is γ'. D' intersects B_i in a disk D'' with $\partial D'' = \gamma'$. The annulus A' is a product annulus and it is contained in the region homeomorphic to $S_i \times I$. Thus the boundary components of A' are disjoint from the cores of the 2-handles attached to S_i. In particular the boundary γ' of D'' is disjoint from the cores of the 2-handles attached to S_i. Then D'' is a compression disk for S_i contained in B_i whose boundary is disjoint from a set of compressing disks contained in A_i, this fact contradicts the weakly compressibility of S_i. Therefore γ must bound a disk in $\bar{S}_i$. Push the disk γ bounds in $\bar{S}_i$ slightly into A_i, this is a disk D in A_i whose boundary is parallel to γ in the component of R_i adjacent to C across γ. Replacing the subdisk of R_i bounded by γ by the disk D allows us to remove the nested curves.

Thus $S_i \cap R_i$ contains one arc and non nested curves.

Let R_{A_i} denote $R_i \cap A_i$ and R_{B_i} denote $R_i \cap B_i$. In R_{B_i} there are non nested closed curves that bound disks in A_i. We can assume that $R_{A_i} \cap S_i$ is empty, otherwise we must see nested curves.

R_{B_i} is a planar surface contained in B_i. R_{B_i} is incompressible, for otherwise, by doing a compression we get a surface A_i' isotopic to A_i with fewer intersections with S_i. However R_{B_i} must be ∂-compressible. This can be seen by looking at the intersections of R_{B_i} with a collection of meridian disks and spanning annuli in B_i. There are 4 types of ∂-compressions, determined by the types of arcs shown in Figure 5.

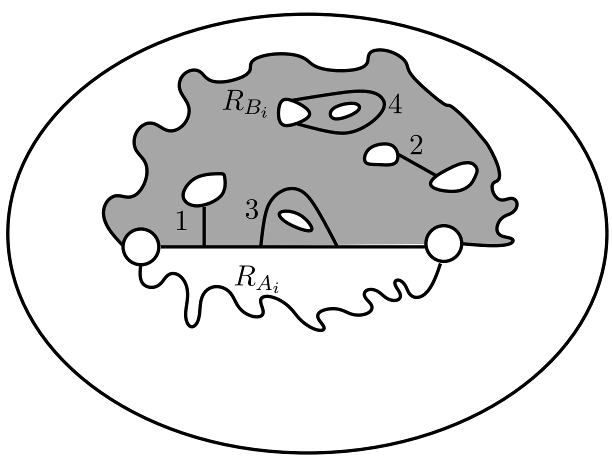


Figure 5. Four types of arcs

If Δ is a ∂-compressing disk for R_{B_i} where $\partial\Delta = \delta_1 \cup \delta_2$, $\delta_1 \subset R_{B_i}$, $\delta_2 \subset S_i$, then a boundary compression along Δ pushes a regular neighborhood of δ_1 into A_i. After performing boundary compressions corresponding to arcs of type 1 and 2 the number of curves of intersection is decreased by 1. By performing boundary compressions corresponding to arcs of type 3 and 4, nested curves are generated, which can be removed. See Figure 6.

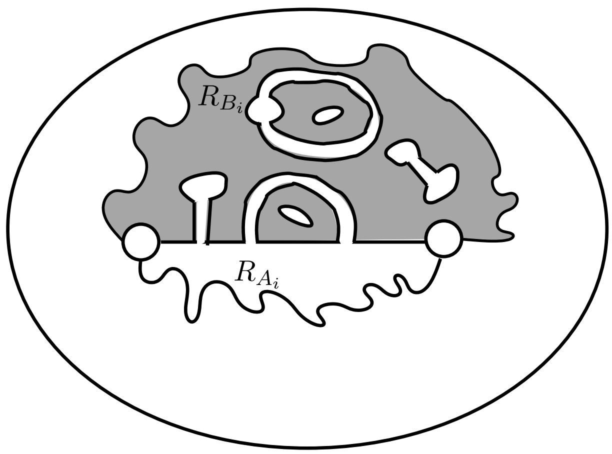


Figure 6. R_{B_i} after boundary compressing

Thus $R_i \cap S_i$ contains only one arc. Therefore the annulus A intersects each $S_i \in \mathcal{S}$ in one arc.

$\square$

This proposition allows us to push 1-handles and 2-handles away from the annulus A. Moreover a collection of 1-handles N_i (or a collection of 2-handles T_i) can be pushed away from A in such a way that N_i (or T_i) is totally contained in $E(K_j) \cap E(K_1 \sharp K_2)$, for some $j = 1, 2$.

In other words, a circular (locally) thin decomposition for $E(K_1 \sharp K_2)$ induces circular locally thin decompositions for $E(K_1)$ and $E(K_2)$.

Corollary 5.3. *Suppose $K = K_1 \sharp K_2$ is in circular (locally) thin position. Let $E(K) = (F \times I) \cup N_1 \cup T_1 \cup ... \cup N_m \cup T_m / F \times -1 \sim F \times 1$ be a handle decomposition realizing a circular (locally) thin position. Let $\mathcal{N}$ be the collection of N_i's, let $\mathcal{T}$ be the collection of T_i's. Then there are subcollections $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{N}$ such that $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{N}$ and $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$, and subcollections $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathcal{T}$ such that $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{T}$ and $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$, such that $\mathcal{N}_i$ and $\mathcal{T}_i$ define a circular handle decomposition for $E(K_i)$, $i = 1, 2$.*

Proof. By Proposition 5.1 the annulus $A = \Sigma \cap E(K_1 \sharp K_2)$ intersects each F_i and each S_i in a single arc. Each arc $A \cap F_i$ separates F_i in F_i'(contained in $E(K_1)$) and F_i'' (contained in $E(K_2)$), and each arc $A \cap S_i$ separates S_i in S_i' (contained in $E(K_1)$) and S_i'' (contained in $E(K_2)$). If one 1-handle of N_i is attached to $F_i' \times I$ and a 2-handle of T_i is attached to $S_i'' \times I$ (or viceversa), then these handles determine compressing disks for S_i which are disjoint and lie on opposite sides of S_i, which contradicts that S_i is weakly incompressible. Therefore the collection of 1-handles N_i must be attached either to $F_i' \times I$ or to $F_i'' \times I$, say $F_i' \times I$, and then the collection of 2-handles T_i is attached along S_i'.

Since $F_i \simeq F_i' \sharp F_i''$, if the collection N_i has been attached along F_i' (or F_i'') then the surface S_i is homeomorphic to $S_i' \sharp F_i''$ (or $F_i' \sharp S_i''$).

In general we will see the following: Begin with the surface $F_1 = F \simeq F_1' \sharp F_1''$, we will see a subcollection $\mathcal{N}_j^1$ of $\mathcal{N}$ and a subcollection $\mathcal{T}_j^1$ of $\mathcal{T}$ contained in $E(K_j)$, say $j = 1$. In other words there is i_0 $(1 \leq i_0 \leq m)$ such that for every $1 \leq i \leq i_0$, the handles N_i are attached along F_i' and the handles T_i are attached along S_i'.

If $i_0 = m$ then $\mathcal{N}$ and $\mathcal{T}$ happened to be contained, say in $E(K_1)$, then the knot K_2 is fibered and $\mathcal{N}_1 = \mathcal{N}$, $\mathcal{N}_2 = \emptyset$, $\mathcal{T}_1 = \mathcal{T}$, $\mathcal{T}_2 = \emptyset$.

If $i_0 < m$ then there is a subcollection $\mathcal{N}_2^1$ of $\mathcal{N} - \mathcal{N}_1^1$ and a subcollection $\mathcal{T}_2^1$ of $\mathcal{T} - \mathcal{T}_1^1$ contained in $E(K_2)$. In other words there is i_1 $(i_0 + 1 \leq i_1 \leq m)$ such that for every $i_0 + 1 \leq i \leq i_1$, the handles N_i are attached along F_i'' and the handles T_i are attached along S_i''.

If $i_1 < m$ then there is a subcollection $\mathcal{N}_1^2$ of $\mathcal{N} - (\mathcal{N}_1^1 \cup \mathcal{N}_2^1)$ and a subcollection $\mathcal{T}_1^2$ of $\mathcal{T} - (\mathcal{T}_1^1 \cup \mathcal{T}_2^1)$ contained in $E(K_1)$. In other words there is i_2 $(i_1 + 1 \leq i_2 \leq m)$ such that for every $i_1 + 1 \leq i \leq i_2$, the handles N_i are attached along F_i' and the handles T_i are attached along S_i'.

We conclude when $i_s = m$, then we have a subcollection $\mathcal{N}_j^s$ of $\mathcal{N} - (\mathcal{N}_1^1 \cup \mathcal{N}_2^1 \cup \mathcal{N}_1^2 \cup ... \cup \mathcal{N}_{j' \neq j}^{s-1})$ and a subcollection $\mathcal{T}_j^s$ of $\mathcal{T} - (\mathcal{T}_1^1 \cup \mathcal{T}_2^1 \cup \mathcal{T}_1^2 \cup ... \cup \mathcal{T}_{j'}^{s-1})$ contained in $E(K_j)$, where $j, j' \in \{1, 2\}$ and $j \neq j'$.

Thus the subcollection $\mathcal{N}_j$ is given by $\cup \mathcal{N}_j^k$ and the subcollection $\mathcal{T}_j$ is given by $\cup \mathcal{T}_j^k$, for $j = 1, 2$. This proves the corollary. □

Remark 5.4. It is not hard to see that we can rearrange the collections of 1-handles and 2-handles in such a way that we first glue all handles contained in one summand, say $E(K_1)$, and then all the handles contained in $E(K_2)$.

The following result is an immediate consequence:

Corollary 5.5. *If $K = K_1\sharp K_2$ is almost fibered then either K_1 or K_2 is fibered, say K_1, and K_2 is not fibered.*

Now we are ready to prove:

Theorem 5.6. *Let K_1 and K_2 be knots in S^3. The equation*
$cw(E(K_1\sharp K_2)) = cw(E(K_1))\sharp cw(E(K_2))$
holds for the following cases:

1. *K_1 and K_2 are fibered knots.*
2. *K_1 is fibered and K_2 is not fibered.*
3. *K_1 and K_2 are non-fibered knots. $E(K_1)$ and $E(K_2)$ have circular thin positions containing minimal genus Seifert surfaces as a thin level.*

Proof. (1) Easily follows from the well known fact that connected sum of two fibered knots is fibered.

(2) Let F be the fiber for $E(K_1)$ and assume $E(K_2)$ is in circular thin position with $\{R_i\}_1^n$ the collection of thin levels and $\{S_i\}_1^n$ the collection of thick levels. Then $E(K_1\sharp K_2)$ inherits a circular handle decomposition with thin levels homeomorphic to the collection $\{F\sharp R_i\}_1^n$ and thick levels homeomorphic to $\{F\sharp S_i\}$, such decomposition has circular width given by $cw(E(K), D) = \{c(F\sharp S_i)\}_1^n$ modulo non-increasing order.

Suppose that $E(K_1\sharp K_2)$ has a circular a circular thin decomposition with thin levels $\{T_j\}_1^m$ and thick levels $\{U_j\}_1^m$. Proposition 5.1 together with the assumption that K_1 is fibered imply that $T_j = F\sharp T_j'$ and $U_j = F\sharp U_j'$, inducing a circular handle decomposition on $E(K_2)$ with thick levels $\{U_j'\}$. Such decomposition has circular width $cw(E(K_2), D') = \{c(U_j'\}_1^m$ modulo non-increasing order. Thus we have the following inequality:

$$\{c(S_i)\} \leq \{c(U_j')\} \tag{3}$$

modulo non-increasing order.

If we add $c(F)$ to both sides of equation (3) we obtain:

$$\{c(S_i) + c(F)\} = \{c(F\sharp S_i)\} \leq \{c(U_j') + c(F)\} = \{c(U_j)\}$$

modulo non-increasing order, which is equivalent to:

$$cw(E(K_1))\sharp cw(E(K_2)) \leq cw(E(K_1\sharp K_2))$$

.

(3) Let D_1 be a circular handle decomposition for $E(K_1)$ which realizes $cw(E(K_1))$. Let $\{F_i\}_{i=1}^k$ be the collection of thin levels and $\{G_i\}_{i=1}^k$

be the collection of thick levels for D_1. Then $cw(E(K_1)) = \{c(G_i)\}_{i=1}^{k}$ modulo non-increasing order.

Let D_2 be a circular handle decomposition for $E(K_2)$ which realizes $cw(E(K_2))$. Let $\{R_j\}_{j=1}^{l}$ be the collection of thin levels and $\{S_j\}_{j=1}^{l}$ be the collection of thick levels for D_2. Then $cw(E(K_2)) = \{c(S_j)\}_{j=1}^{l}$ modulo non-increasing order.

Assume that F_1 and R_1 are minimal genus Seifert surfaces for K_1 and K_2, respectively.

We know that D_1 and D_2 induce a circular locally thin decomposition D on $E(K_1 \sharp K_2)$ with circular width given by:

$$cw(E(K_1 \sharp K_2), D) = cw(E(K_1)) \sharp cw(E(K_2)) = \{c(G_i \sharp R_1)\} \cup \{c(F_1 \sharp S_j)\}$$

Modulo non-increasing order.

Moreover we know that

$cw(E(K_1 \sharp K_2)) \leq cw(E(K_1)) \sharp cw(E(K_2))$.

In order to prove that the equality holds we need to show that $cw(E(K_1)) \sharp cw(E(K_2)) \leq cw(E(K_1 \sharp K_2))$.

Suppose that D' is a circular decomposition for $E(K_1 \sharp K_2)$ which realizes $cw(E(K_1 \sharp K_2))$. Let $\{T_i\}_{i=1}^{m}$ be the collection of thin levels and $\{U_i\}_{i=1}^{m}$ be the collection of thick levels for D'. Then $cw(E(K_1 \sharp K_2)) = \{c(U_i)\}_{i=1}^{m}$ modulo non-increasing order.

By Proposition 5.1 each T_i and U_i is homeomorphic to a boundary connected sum of Seifert surfaces. Using Corollary 5.3 we can find $s \in \{1, 2, ..., m\}$ such that:

$$\begin{array}{ll}
T_1 \simeq T_1' \sharp T_1'' & \\
U_i \simeq T_1' \sharp U_i'' & 1 \leq i < s \\
T_i \simeq T_1' \sharp T_i'' & 1 \leq i < s \\
T_s \simeq T_1 \sharp T_1'' & \\
U_i \simeq U_i' \sharp T_1'' & s \leq i \leq m \\
T_i \simeq T_i' \sharp T_1'' & s < i < m \\
T_m = T_1. &
\end{array}$$

Thus $E(K_1)$ inherits a circular decomposition D_1' with thin levels $\{T_1'\} \cup \{T_i'; s < i < m\}$ and thick levels $\{U_i' : s \leq i \leq m\}$. Then $cw(E(K_1), D_1') = \{c(U_i')\}$ modulo non-increasing order.

Also $E(K_2)$ inherits a circular decomposition D_2' with thin levels $\{T_1''\} \cup \{T_i''; 1 < i < s\}$ and thick levels $\{U_i'' : 1 \leq i < s\}$. Then $cw(E(K_2), D_2') = \{c(U_i'')\}$ modulo non-increasing order.

Also we know;

$$cw(E(K_1)) \leq cw(E(K_1), D_1') \qquad cw(E(K_2)) \leq cw(E(K_2), D_2') \tag{4}$$

The following equations are true as well

$$c(U_i') + c(T_1'') = c(U_i) \qquad c(U_j'') + c(T_1') = c(U_j) \tag{5}$$

$$c(F_1) \leq c(T_1') \qquad c(R_1) \leq c(T_1'') \tag{6}$$

Remember that F_1 is a minimal genus Seifert surface for K_1 and R_1 is a minimal genus Seifert surface for K_2.

Equations (4), (5) and (6) imply the following:

$$\{c(G_i) + c(R_1)\} \leq \{c(G_i) + c(T_1'')\} \leq \{c(U_i') + c(T_1'')\} = \{c(U_i)\} \tag{7}$$

and

$$\{c(S_j) + c(F_1)\} \leq \{c(S_i) + c(T_1')\} \leq \{c(U_j'') + c(T_1')\} = \{c(U_j)\} \tag{8}$$

Modulo non-increasing order.

Notice that ;

$$\{c(G_i) + c(R_1)\} \cup \{c(S_j) + c(F_1)\} = cw(E(K_1))\sharp cw(E(K_2))$$

and

$$\{c(U_i)\} \cup \{c(U_j)\} = cw(E(K_1 \sharp K_2)).$$

Then applying Proposition 4.4 to the lefthand side and righthand side of equations (7) and (8) , we obtain

$$cw(E(K_1))\sharp cw(E(K_2)) \leq cw(E(K_1 \sharp K_2))$$

This proves the theorem. $\square$

The following question remains open;

Question 5.7. Does a knot in circular thin position contain a minimal genus Seifert as a thin surface?

There is evidence that a minimal genus Seifert surface appears in a circular thin position. All non fibered knots up to ten crossings are almost fibered and the thin surface appearing in the circular thin decomposition is of minimal genus.

If the answer to the question is affirmative, then the additivity of circular width would be true in general.

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